ISSN 2349-7815

International Journal of Recent Research in Electrical and Electronics Engineering (IJRREEE)
Vol. 11, Issue 4, pp: (6-10), Month: October - December 2024, Available at: www.paperpublications.org

Evaluating Fractional Derivatives of Two Types of Matrix Fractional Functions

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DOI: https://doi.org/10.5281/zenodo.14160553

Published Date: 14-November-2024

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we evaluate arbitrary order fractional derivative of two types of matrix fractional functions. In fact, our results are generalizations of traditional calculus results.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, matrix fractional functions.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, mechanics, electrical engineering, viscoelasticity, biology, control theory, dynamics, economics, and other fields [1-16].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, Jumarie's modified R-L fractional derivative [17-21]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we study the fractional differential problem of the following two types of matrix fractional functions:

$$cos_{\alpha}\left(E_{\alpha}(tAx^{\alpha})\right),$$

$$sin_{\alpha}\left(E_{\alpha}(tAx^{\alpha})\right),$$

where $0 < \alpha \leq 1$, t is a real number, and A is a real matrix. Using some techniques, we can evaluate any order fractional derivative of these two types of matrix fractional functions. In fact, our results are generalizations of traditional calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([22])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \quad (1)$$

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where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([23])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Definition 2.3 ([24])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([25])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([26])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{8}$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2n}, \tag{9}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2n+1)}. \tag{10}$$

ISSN 2349-7815

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Definition 2.6 ([27]): If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \quad (11)$$

And the matrix α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (12)$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \quad (13)$$

III. MAIN RESULTS

In this section, we evaluate arbitrary order fractional derivative of two types of matrix fractional functions.

Theorem 3.1: *If* $0 < \alpha \leq 1$, t *is a real number*, m *is a positive integer, and* A is a real matrix, *then*

$$\left({}_0D_x^\alpha\right)^m\left[cos_\alpha\left(E_\alpha(tAx^\alpha)\right)\right] = t^m A^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2n)^m E_\alpha(2ntAx^\alpha), \quad (14)$$

and

$$\left({}_0D_x^\alpha\right)^m\left[sin_\alpha\left(E_\alpha(tAx^\alpha)\right)\right] = t^m A^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)^m E_\alpha((2n+1)tAx^\alpha). \quad (15)$$

Proof

$$\left({}_0D_x^\alpha\right)^m\left[cos_\alpha\left(E_\alpha(tAx^\alpha)\right)\right]$$
$$= \left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(E_\alpha(tAx^\alpha)\right)^{\otimes_\alpha 2n}\right]$$
$$= \left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}E_\alpha(2ntAx^\alpha)\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left({}_0D_x^\alpha\right)^m\left[E_\alpha(2ntAx^\alpha)\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2nt)^m A^m E_\alpha(2ntAx^\alpha)$$
$$= t^m A^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2n)^m E_\alpha(2ntAx^\alpha).$$

And

$$\left({}_0D_x^\alpha\right)^m\left[sin_\alpha\left(E_\alpha(tAx^\alpha)\right)\right]$$
$$= \left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(E_\alpha(tAx^\alpha)\right)^{\otimes_\alpha (2n+1)}\right]$$
$$= \left({}_0D_x^\alpha\right)^m\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}E_\alpha((2n+1)tAx^\alpha)\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left({}_0D_x^\alpha\right)^m\left[E_\alpha((2n+1)tAx^\alpha)\right]$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}((2n+1)t)^m A^m E_\alpha((2n+1)tAx^\alpha)$$
$$= t^m A^m \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)^m E_\alpha((2n+1)tAx^\alpha).$$

IV. CONCLUSION

In this paper, we find arbitrary order fractional derivative of two types of matrix fractional functions. In fact, our results are generalizations of classical calculus results. In the future, we will continue to use our methods to solve problems in applied mathematics and fractional differential equations.

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